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Non-classical analysis and Painlevé property for the Kupershmidt equations

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Abstract. By means of the non-classical Lie approach and the direct method, all the similarity reductions of the Kupershmidt equations are found. Two reduction equations which can also be obtained by the classical Lie approach are the Painlevé type II and IV respectively. The other three reduction equations which cannot be obtained by the classical Lie approach are only linear or Riccati type.

1. Introduction

The equations governing the propagation of long waves in shallow water consist of pair of coupled first-order partial differential equations (PDEs) which can be interpreted as a Hamiltonian system in several different ways [1]. One remarkable model is the so-called Kupershmidt equations

$$\Delta_1^{(2)} \equiv u_t + uu_x + h_x + Cu_{xx} = 0 \tag{1}$$

$$\Delta_2^{(2)} \equiv h_t + uh_x + hu_x - Ch_{xx} = 0 \tag{2}$$

which admit the tri-Hamiltonian structure and the infinitely many conservation laws [2], where subscripts are partial differentiations.

The symmetry reduction method is a powerful tool to seek the solutions of PDEs. The standard method for finding similarity reductions of a given PDE is to use the classical Lie approach [3–5], Fahrunisa Neyzi and Yavuz Nutku [5] had used the method to reduce (1) and (2) and pointed out that Kuperschmidt equation is of Painlevé type II. Kawamoto, Paquin and Winternitz had also used the method and get a Painlevé IV type reduction [6]. Recently, Clarkson and Kruskal developed a direct and simple method to reduce a given PDE to some ordinary differential equations (ODEs) [7–9] or some PDEs in lower dimensions [10]. Furthermore, Levi and Winternitz [11] had pointed out that all the similarity reductions obtained by the direct method for the single PDE cases can also be obtained by the 'nonclassical method' due to Bluman and Cole [12].

In section 2 of this paper, we will extend the direct method to the PDEs system case: Kuperschmidt equations (1) and (2). Five types of the similarity solutions are obtained. The first two of them are just the Painlevé type II and IV reductions obtained by the classical Lie approach and the other three types of reduction equation are only linear or Riccati type equations. The group theoretical interpretation for the latter three types of reduction equations is given in section 3. Section 4 is a summary and discussions.

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2. Symmetry reductions of the Kupershmidt equations by the direct method

Analogous to the application of the direct method to the single PDE cases [7–10] we can easily prove that it is sufficient to seek the similarity reduction of the Kupershmidt equations in the special forms

$$u(x, t) = \alpha(x, t) + \beta(x, t) w(Z(x, t)) \tag{3}$$

$$h(x, t) = A(x, t) + B(x, t) Q(Z(x, t)). \tag{4}$$

Substituting (3) and (4) into (1) gives

$$\begin{aligned} C\beta Z_x^2 w'' &+ (\beta Z_t + \alpha\beta Z_x + 2C\beta_x Z_x + C\beta Z_{xx}) w' + (\beta_t + \alpha\beta_x + \beta\alpha_x + C\beta_{xx}) w \\ &+ \beta\beta_x w^2 + \beta^2 Z_x w w' + BZ_x Q' + B_x Q + \alpha_t + \alpha\alpha_x + A_x + C\alpha_{xx} = 0 \end{aligned} \tag{5}$$

$$\begin{aligned} A\beta Z_x w' &+ (A\beta)_x w - CBZ_x^2 Q'' + (BZ_t + \alpha BZ_x - 2CB_x Z_x - CBZ_{xx}) Q' \\ &+ (B_t + B\alpha_x + \alpha B_x - CB_{xx}) Q + (B\beta)_x Q w + B\beta Z_x (Qw)' \\ &+ A_t + \alpha_x A + \alpha A_x - CA_{xx} = 0 \end{aligned} \tag{6}$$

where primes are derivatives with respect to Z.

Equations (5) and (6) are ODEs of w and Q only for the ratios of the coefficients of different derivatives and powers of w and H being functions of Z. That is to say that the following constrained conditions for $Z_x \neq 0$ must be satisfied:

$$\beta Z_t + \alpha\beta Z_x + 2C\beta_x Z_x + C\beta Z_{xx} = \beta Z_x^2 \Gamma_1(Z) \tag{7}$$

$$\beta_t + \alpha\beta_x + \alpha_x\beta + C\beta_{xx} = \beta Z_x^2 \Gamma_2(Z) \tag{8}$$

$$\beta\beta_x = \beta Z_x^2 \Gamma_3(Z) \tag{9}$$

$$\beta^2 Z_x = \beta Z_x^2 \Gamma_4(Z) \tag{10}$$

$$BZ_x = \beta Z_x^2 \Gamma_5(Z) \tag{11}$$

$$B_x = \beta Z_x^2 \Gamma_6(Z) \tag{12}$$

$$\alpha_t + \alpha\alpha_x + A_x + C\alpha_{xx} = \beta Z_x^2 \Gamma_7(Z) \tag{13}$$

$$A\beta Z_x = BZ_x^2 \Gamma_8(Z) \tag{14}$$

$$(A\beta)_x = BZ_x^2 \Gamma_9(Z) \tag{15}$$

$$BZ_t + \alpha BZ_x - 2CB_x Z_x - CBZ_{xx} = BZ_x^2 \Gamma_{10}(Z) \tag{16}$$

$$B_t + B\alpha_x + \alpha B_x - CB_{xx} = BZ_x^2 \Gamma_{11}(Z) \tag{17}$$

$$B\beta_x + B_x\beta = BZ_x^2 \Gamma_{12}(Z) \tag{18}$$

$$B\beta Z_x = BZ_x^2 \Gamma_{13}(Z) \tag{19}$$

$$A_t + A\alpha_x + \alpha A_x - CA_{xx} = BZ_x^2 \Gamma_{14}(Z) \tag{20}$$

where $\Gamma_i(Z)$ $(i = 1, 2, \ldots, 14)$ are some arbitrary functions of Z to be determined. Similar to the single PDE cases [7–10], solving (7)–(20), we get the only possible independent similarity reduction for $Z_x \neq 0$:

$$u = -\frac{1}{\theta}(\theta_t x + \lambda_t) + \theta w(Z) \tag{21}$$

$$h = \theta^2 Q(Z) \qquad Z = \theta x + \lambda \tag{22}$$

$$\theta_t = D_0 \theta^3 \tag{23}$$

$$\lambda_{tt} = 2D_0\theta^2\lambda_t + D_0^2\theta^4\lambda - C_1\theta^4 \tag{24}$$

$$Cw'' + ww' + Q' - D_0^2 Z + C_1 = 0 \tag{25}$$

$$-CQ'' + D_0 Q + (Qw)' = 0 \tag{26}$$

with D_0 and C_1 being arbitrary constants. There exist two subcases for further discussions: (a) $D_0 = 0$ and (b) $D_0 = \text{constant} \neq 0$.

(i) When $D_0 = 0$, (21)-(26) become

$$u = -\frac{1}{D_1}(D_2 - D_1^4 t) + D_1 w(Z) \tag{27}$$

$$h = D_1^2 Q(Z) \qquad Z = D_1 x + (-\tfrac{1}{2}D_1^4 t^2 + D_2 t + D_3) \tag{28}$$

$$Q = -f_1 - C_1 Z - \tfrac{1}{2}w^2 - Cw' \tag{29}$$

and w satisfies the ODE:

$$C^2 w''' - (C_1 Z + f_1)w' - \tfrac{3}{2}w^2 w' - C_1 w = 0 \tag{30}$$

where D_1, D_2, D_3 and f_1 are all arbitrary constants. The reduction equation (30) is just the result obtained by the classical Lie approach. Taking the transformations

$$w(Z) = 2C_1^{1/3}C^{1/3}w_1(\xi) \qquad \xi = C_1^{1/3}C^{-2/3}Z + f_1 C_1^{-1} \tag{31}$$

for equation (30) leads to the Painlevé II equation

$$w_1'' = w_1^3 + 2\xi w_1 + f_2 \tag{32}$$

with f_2 an another integral constant.

(ii) When $D_0 = \text{constant} \neq 0$, the symmetry reduction equations become

$$u = -\frac{D_0}{2(D_1 - D_0 t)}x + \frac{\sqrt{2}}{2}D_0\lambda_2 - \frac{\sqrt{2}}{2}D_0\lambda_1\frac{1}{D_1 - D_0 t} + \frac{w(Z)}{[2(D_1 - D_0 t)]^{1/2}} \tag{33}$$

$$h = \frac{Q(Z)}{2(D_1 - D_0 t)} \tag{34}$$

$$Z = -\frac{x}{[2(D_1 - D_0 t)]^{1/2}} + \frac{\lambda_1}{(D_1 - D_0 t)^{1/2}} + \lambda_2(D_1 - D_0 t)^{1/2} + \frac{C_1}{D_0^2} \tag{35}$$

$$Q = -Cw' - \tfrac{1}{2}w^2 + \tfrac{1}{2}D_0^2 Z^2 - C_1 Z - f_1 \tag{36}$$

and

$$C^2 w''' + (\tfrac{1}{2}D_0^2 Z^2 - C_1 Z - D_0 C - f_1)w' - \tfrac{1}{2}D_0 w^2 - \tfrac{3}{2}w^2 w' + (D_0^2 Z - C_1)w$$
$$+ \tfrac{1}{2}D_0^3 Z^2 - D_0 C_1 Z - D_0^2 C - D_0 f_1 = 0 \tag{37}$$

where D_1, λ_1, λ_2, and f_1 are all arbitrary constants. By means of the following transformations

$$w=\sqrt{2|CD_0|}\,(x+y(x)) \qquad x=-\frac{D_0}{\sqrt{2|CD_0|}}(Z-C_1/D_0^2) \tag{38}$$

equation (37) becomes the Painlevé type IV equation

$$\frac{d^2y}{dx^2}=\frac{1}{2y}\left(\frac{dy}{dx}\right)^2+\frac{3}{2}y^3+4xy^2+2(x^2-\alpha)y+\frac{\beta}{y} \tag{39}$$

where β is an integral constant and

$$\alpha=\frac{1}{2|D_0C|}\left[\frac{C_1^2}{2D_0^2}-D_0C-f_1\right]. \tag{40}$$

When constants C_1 and λ_2 are taken as zero, this type of reduction is just that obtained by Paquin and Winternitz [6].

Similarly, in $Z_x=0$ case, we can also get only three independent similarity solutions.

(iii) In the first solution, we have

$$u=w(t) \tag{41}$$

$$h=\Gamma_4x+Q(t) \tag{42}$$

$$w'+\Gamma_4=0 \tag{43}$$

$$Q'+\Gamma_4w=0 \tag{44}$$

with Γ_4 being an arbitrary constant. Now the reduced ODEs of w and Q equations are only linear equations. The general solutions of (43) and (44) read

$$w=-\Gamma_4t+d_1 \tag{45}$$

$$Q=\tfrac{1}{2}\Gamma_4^2t^2-\Gamma_4d_1t+d_2 \tag{46}$$

where d_1 and d_2 are two integral constants.

(iv) In the second case, we have

$$u=-C_1+(x+C_1t+C_2)w(t) \tag{47}$$

$$h=Q(t) \tag{48}$$

$$w'+w^2=0 \tag{49}$$

$$Q'+Qw=0. \tag{50}$$

In this case the reduction equations of w is a simple Riccati equation and the Q equation (50) is only a linear one. Their general solutions are

$$w=\frac{1}{t+t_0} \tag{51}$$

$$Q=\frac{C_3}{t+t_0} \tag{52}$$

where C_1, C_2 and C_3 are also constants of integration.

(v) In the third case, we have

$$u=\frac{1}{t+t_0}x+w(t) \tag{53}$$

$$h=\frac{C_1}{(t+t_0)^2}x+Q(t) \tag{54}$$

$$w'+\frac{w}{(t+t_0)}+\frac{C_1}{(t+t_0)^2}=0 \tag{55}$$

$$Q'+\frac{Q}{(t+t_0)}+\frac{C_1 w}{(t+t_0)^2}=0. \tag{56}$$

In this case, the w and Q equations are all linear equations with the general solutions

$$w=\frac{w_0}{t+t_0}-\frac{C_1\ln(t+t_0)}{t+t_0} \tag{57}$$

$$Q=\frac{Q_0}{t+t_0}+\frac{w_0C_1}{(t+t_0)^2}-\frac{C_1^2}{(t+t_0)^2}[\ln(t+t_0)+1] \tag{58}$$

where w_0, Q_0, C_1 and t_0 are all arbitrary constants.

3. Similarity reductions from the non-classical method

In order to use the 'non-classical method' of Bluman and Cole [12], we insert *two* conditional constrained equations at first

$$\Delta_1^{(1)}\equiv Tu_t+Xu_x-U=0 \tag{59}$$

and

$$\Delta_2^{(1)}\equiv Th_t+Xh_x-H=0 \tag{60}$$

where T, X, H and U are some undetermined functions of x, t, u and h. Now we apply the standard algorithm that provides the symmetry algebra, i.e. the Lie algebra of the Lie group of local point transformations leaving the joint solutions set of equations (1), (2), (59) and (60) invariant. The vector field has the form

$$V=T\frac{\partial}{\partial t}+X\frac{\partial}{\partial x}+U\frac{\partial}{\partial u}+H\frac{\partial}{\partial h} \tag{61}$$

where T, X, U and H are the same as in (59) and (60). The algorithm starts by constructing the prolongation of the vector field V, i.e. differential operator of the form

$$\mathrm{pr}^{(2)}V=V+\{U_x\}\frac{\partial}{\partial u_x}+\{U_t\}\frac{\partial}{\partial u_t}+\{H_x\}\frac{\partial}{\partial h_x}+\{H_t\}\frac{\partial}{\partial h_t}+\{U_{xx}\}\frac{\partial}{\partial u_{xx}}+\{H_{xx}\}\frac{\partial}{\partial h_{xx}} \tag{62}$$

where the functions $\{U_x\}$, $\{U_t\}$, $\{U_{xx}\}$, $\{H_x\}$, $\{H_t\}$ and $\{H_{xx}\}$ are the first and second extensions. They can be expressed in terms of T, X, U, H and their derivatives. For instance,

$$\{H_x\}=H_x+H_uu_x+H_hh_x-h_x(X_x+X_uu_x+X_hh_x)-h_t(T_x+T_uu_x+T_hh_x) \tag{63}$$

$$\{H_t\}=H_t+H_uu_t+H_hh_t-h_x(X_t+X_uu_t+X_hh_t)-h_t(T_t+T_uu_t+T_hh_t) \tag{64}$$

$$\{H_{xx}\} = H_{xx} + 2H_{xu}u_x + 2H_{xh}h_x + H_{uu}u_x^2 + 2H_{uh}h_xu_x + H_uu_{xx} + H_{hh}h_x^2 + H_hh_{xx}$$
$$- 2h_{xx}(X_x + X_uu_x + X_hh_x) - 2h_{xt}(T_x + T_uu_x + T_hh_x)$$
$$- h_xX_{xx} + 2X_{xu}u_x + 2X_{xh}h_x + X_{uu}u_x^2 + 2X_{uh}u_xh_x + X_{hh}h_x^2 + X_uu_{xx} + X_hh_{xx})$$
$$- h_t(T_{xx} + 2T_{xu}u_x + 2T_{xh}h_x + T_{uu}u_x^2 + 2T_{uh}u_xh_x + T_{hh}h_x^2 + T_uu_{xx} + T_hh_{xx}). \tag{65}$$

Then applying the prolongation to the *four* equations, (1), (2), (59) and (60), yields

$$\mathrm{pr}^{(2)}\Delta_i^{(2)}|_{\Delta_i^{(2)}=0,\Delta_j^{(1)}=0} = 0 \qquad i = 1, 2 \qquad j = 1, 2 \tag{66}$$

$$\mathrm{pr}^{(1)}\Delta_j^{(1)}|_{\Delta_j^{(1)}=0,\Delta_i^{(2)}=0} = 0 \qquad i = 1, 2 \qquad j = 1, 2. \tag{67}$$

Equations (67) are satisfied identically while equations (66) lead to a set of determining equations that must be solved. For further discussions we consider two different cases to write down and partially solve the equations: $T \neq 0$ and $T = 0$.

Case 1: when $T \neq 0$, we can put $T = 1$ with no loss of generality. After insert the ansatz

$$X = \delta(t)x + \Delta(t) \tag{68}$$

$$H = \alpha(t)h \tag{69}$$

$$U = \beta(t)u + \gamma(t) \tag{70}$$

into the determining equation system, we get

$$\alpha = -2\delta(t) \qquad \beta = -\delta(t) \tag{71}$$

$$\delta_t + 2\delta^2 = 0 \tag{72}$$

$$\gamma_t + 2\gamma\delta = 0 \tag{73}$$

$$\Delta_t + 2\Delta\delta - \gamma = 0. \tag{74}$$

To solve (72)-(74) there exist two subcases.

Case 1_a: $\delta = 0$, $\gamma = \gamma_0$, $\Delta = \gamma_0 t + \Delta_0$ with γ_0 and Δ_0 being arbitrary constants, i.e. the vector fields have the form

$$V = \frac{\partial}{\partial t} + (\gamma_0 t + \Delta_0)\frac{\partial}{\partial x} + \gamma_0\frac{\partial}{\partial u}. \tag{75}$$

After solving the Lagrange conditions

$$\frac{\mathrm{d}t}{T} = \frac{\mathrm{d}x}{X} = \frac{\mathrm{d}u}{U} = \frac{\mathrm{d}h}{H} \tag{76}$$

we re-obtained the first type of reduction solution given in case (i) of the last section.

Case 1_b:

$$\delta = \frac{-D_0}{2(D_1 - D_0t)} \tag{77}$$

$$\gamma = \frac{\lambda_1}{D_1 - D_0t} \tag{78}$$

$$\Delta = \frac{\lambda_2}{D_1 - D_0t} - \frac{\lambda_1}{D_0} \tag{79}$$

with D_0, D_1, λ_1 and λ_2 being arbitrary constants, i.e. the vector fields can be written, after multiplying V by $(D_1 - D_0 t)$, as

$$V = (D_1 - D_0 t)\frac{\partial}{\partial t} + \left[\lambda_2 - \frac{1}{2}D_0 x - \frac{\lambda_1}{D_0}(D_1 - D_0 t)\right]\frac{\partial}{\partial x} + D_0 h\frac{\partial}{\partial h} + \left(\frac{D_0}{2}u + \lambda_1\right)\frac{\partial}{\partial u}. \tag{80}$$

Substituting (80) into (76), we can easily find the invariant dependent and independent variables. The final reduction solutions is just the case (ii) obtained by the direct method shown in section 2. In fact, (75) and (80) can also be obtained by the classical Lie approach that is to say, (75) and (80) are also the solutions of the equations

$$\mathrm{Pr}^{(2)}\Delta_i^{(2)}|_{\Delta_i^{(2)}=0} = 0 \qquad i = 1, 2. \tag{81}$$

Case 2: when $T = 0$, we can put $X = 1$ without loss of generality. Similar to $T \neq 0$ case, after putting the ansatz

$$U = \alpha(x, t)u + \beta(x, t) \tag{82}$$

$$H = \delta(t) \tag{83}$$

into the determining equation system, we have

$$\alpha\delta = 0 \tag{84}$$

$$\alpha_x + \alpha^2 = 0 \tag{85}$$

$$\alpha_t + \alpha\beta = 0 \tag{86}$$

$$\beta_t + \beta^2 = 0 \tag{87}$$

$$\beta_x + \alpha\beta = 0 \tag{88}$$

and

$$\delta_t + 2\beta\delta = 0. \tag{89}$$

It is easy to prove that there exist three types of solutions of (84)–(89).

Case 2_a:

$$\alpha = \beta = 0 \qquad \delta = \text{constant} = \Gamma_4 \tag{90}$$

i.e.

$$V = \frac{\partial}{\partial x} + \Gamma_4\frac{\partial}{\partial h}. \tag{91}$$

The corresponding similarity reduction is just the case (iii) of the direct method.

Case 2_b:

$$\delta = 0 \qquad \alpha = \frac{1}{x + C_1 t + C_2} \qquad \beta = \frac{C_1}{x + C_1 t + C_2} \tag{92}$$

where C_1 and C_2 are arbitrary constants. After multiplying the vector field V by $x + C_1 t + C_2$, we get

$$V = (x + C_1 t + C_2)\frac{\partial}{\partial x} + (u + C_1)\frac{\partial}{\partial u}. \tag{93}$$

It is easy to find that after solving (76), the corresponding similarity reduction solution in this case coincides with the case (iv) of the direct method.

Case 2_c:

$$\alpha = 0 \qquad \beta = \frac{1}{t+t_0} \qquad \delta = \frac{C_1}{(t+t_0)^2} \tag{94}$$

where t_0 and C_1 are arbitrary constants. The corresponding vector fields V, multiplying by $(t+t_0)^2$, read

$$V = (t+t_0)^2 \frac{\partial}{\partial x} + (t+t_0) \frac{\partial}{\partial u} + C_1 \frac{\partial}{\partial h} \tag{95}$$

and then the fifth type of the similarity reductions of the direct method which possesses the logarithmic branch points for time t follows immediately.

One can easily prove that (91), (93) and (95) cannot be obtained by the classical Lie approach. That means the vectors shown by (91), (93) and (95) are not the solutions of (81).

To summarize, two types of reductions can be obtained by the classical Lie approach, one is the Painlevé type II and the other is the Painlevé type IV. Three other types of the reductions for the Kupershmidt equations, case (iii) to case (v), can only be obtained by the non-classical Lie approach or the direct method and the reduction equations are only linear and Riccati type.

In the reductions (ii) and (v), the algebraic branch *points* and logarithmic branch *point for time t* (rather than the singular *manifold* for *space time*) are contained respectively though the model possesses the Painlevé property. There exist two different approaches for checking the Painlevé property [13]. The first one is the so-called ARS algorithm established by Ablowitz, Ramani and Segur [14]. This approach requires that all the reductions of a given PDE system must be obtained at first and then check the Painlevé property for all the reduced ODEs. The second one, proposed by Weiss *et al* [15], pointed out that a PDE system will possess the Painlevé property if its solutions are single-valued about a movable singularity manifold $\phi(x, t)$. Furthermore, Kruskal [13, 16] pointed out that the singularity manifold $\phi(x, t)$ may be written as $\phi(x, t) = x + \psi(t)$ with $\psi(t)$ an arbitrary analytic function. In section 2 we see that $\psi(t)$ may be not an analytic function. So, two approaches for checking the Painlevé property of a PDE system are not completely equivalent. It is well known that the crucial point in the implementation of the ARS conjecture resides in obtaining all the reductions of a given PDE system. Fortunately, using the direct method developed by Clarkson *et al* [7–9] and/or the non-classical Lie approach, we can easily get all the reductions of a PDE system.

4. Summary and discussion

In this paper, by using a direct method and the non-classical method, we have obtained all the similarity reductions of the Kupershmidt equations in the special forms (3) and (4) which are equivalent to the most general reduction forms

$$u = U(x, t, w(Z(x, t))) \tag{96}$$

$$h = H(x, t, Q(Z(x, t))). \tag{97}$$

The first two reduction equations which can be obtained by the classical Lie approach are the Painlevé type II and IV respectively. The other reductions, cases (iii), (iv) and (v), have not yet been obtained before. All the reduced ODEs (cases (i)–(v)) possess the Painlevé property in spite of not only the poles but also the algebraic and the logarithmic branch points for time t can enter into the solutions of the Kupershmidt equations (see cases (ii) and (v)). It was pointed out in [17] that the Weiss–Kruskal approach for testing the Painlevé property requires that the singularity manifold is non-characteristic or, in $\phi(x, t) = x + \psi(t)$ $\psi(t)$ should be analytical function of t. But however, from the reduction solutions of the Kupershmidt equation, we see that $\phi(x, t)$ may be characteristic or $\psi(t)$ may be not an analytical function, which indicates that the ARS algorithm should be used to study the Painlevé property of a given PDE system. On other hand, we know also that it is a very difficult task to obtain all the similarity reductions needed in order to apply the ARS algorithm. Fortunately, using the direct method developed recently and used here and the non-classical symmetry reduction method, we can overcome this difficulty. In my opinion, it is not surprising that the algebraic and logarithmic branch points (rather than the arbitrary singular manifold) for time t can be included in the solutions of some integrable models. For instance, the algebraic branch points $\xi = x(\xi t)^{-1/3}$ can be included into the self-similar solutions of the well known integrable model, modified KdV equation [13], which have Painlevé property under the Weiss–Kruskal meaning [15]. The Kupershmidt equations are integrable because of the tri-Hamiltonian structure, and then the infinitely many symmetries and conservation laws are admitted [2]. On the other hand, because the branch points enter into the solutions which may indicate that the processes described by the Kupershmidt equations for fluid system are not invertible.

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